Mechanical Vibration

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Room 4.1.9
Consultations hours - Tuesday 11:15-12:00

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General information - 1st lecture (2020.02.26)

General information

Schedule:

- 15 lectures;
- First lecture - introduction;
- 2nd-14th lecture - short test in the beginning (max. 5 min - non obligatory);
- Last lecture - additional exam.

Passing requirements:

- 13 short tests (non obligatory) - extra points to additional exam (up to 15 points);
- Exam (obligatory) - 50 points (26 points - passing limit).

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General information

Scope of the lecture:

- Analysis and modelling of vibrating system.
- Free vibrations of a single degree of freedom linear systems.
- Harmonically excited vibrations of single degree of freedom linear systems.
- Periodically excited vibrations of single degree of freedom linear systems.
- Phase plane analysis of a linear vibration.
- Phase plane analysis of a nonlinear vibration.
- Free vibration of multi degree of freedom linear systems.
- Excited vibration of multi degree of freedom linear systems.

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General information

Scope of the lecture (continued):

- Vibrations of strings (lateral), rods (longitudinal) and shafts (torsional).
- Lateral free vibrations of beams.
- Lateral excited vibrations of beams.
- Free vibrations of single degree of freedom nonlinear systems.
- Harmonically excited vibrations of single degree of freedom nonlinear systems.
- Parametric and self-excited vibrations.

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Mechanical vibration - general information - 1st lecture (2020.02.26)

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Mechanical vibration - general information

Classification of vibration:

- Deterministic or random;
- Free or excited;
- Damped or undamped;
- Linear or nonlinear:
  - parametric oscillations,
  - self-excited oscillations;
- Vibrations of Single degree of freedom, multi degree of freedom systems or continuous systems;

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Phenomena caused by vibration

Figure 1: Resonant vibration of Tahoma Narrows Bridge

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Phenomena caused by vibration

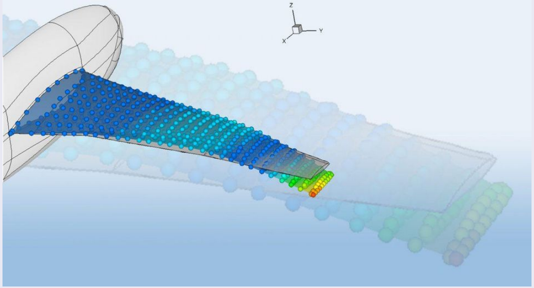

Figure 2: Vibration of a plane wing

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Phenomena caused by vibration

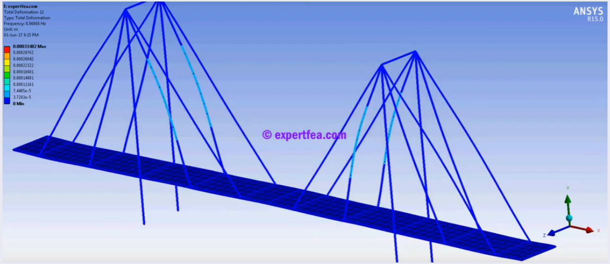

Figure 3: Static deflection a cable-stayed bridge

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Phenomena caused by vibration

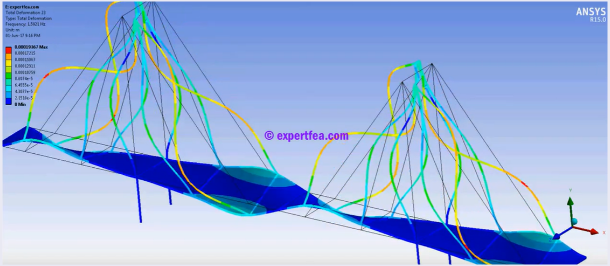

Figure 4: A cable-stayed bridge torsional resonant vibration

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Phenomena caused by vibration

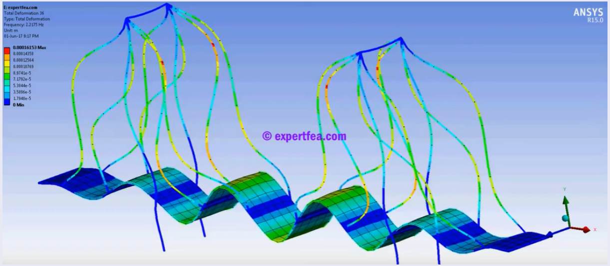

Figure 5: A cable-stayed bridge transverse resonant vibration

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Mechanical vibration - general information

The most relevant phenomena of a vibrational nature:

- fatigue,
- resonant vibration.

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Mechanical vibration - general information

Harmonic process (signal)

$$x(t) = A \cdot \sin(\omega \cdot t + \varphi) \qquad (1)$$

where:

- A - amplitude,
- ω - frequency,
- t - time,
- φ - phase shift.

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Periodicity of signal - 1st lecture (2020.02.26)

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Periodic signal process

Periodical function - definition

$$\forall t \in \mathbb{R} : x(t) = x(t - T) \tag{2}$$

where:

$x(t)$ - signal,

T - period.

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Signal periodicity - example

Case

Determine a period T of the following signal:

$$x(t) = \sin(\omega t) \tag{3}$$

Solution - utilization of the definition

$$\forall t \in \mathbb{R} : \sin(\omega t) = \sin(\omega \cdot (t - T)) \tag{4}$$

$$\omega t = \omega \cdot (t - T_k) + k \cdot 2 \cdot \pi \quad for\ k \in \mathbb{Z} \tag{5}$$

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Signal periodicity - example

Solution - the base period

$$T_k \cdot \omega = k \cdot 2 \cdot \pi \quad for\ k \in \mathbb{Z} \tag{6}$$

$$T_k = \frac{k \cdot 2 \cdot \pi}{\omega} \quad for\ k \in \mathbb{Z} \tag{7}$$

The base period is for k equals $k = 1$:

$$T_o = \frac{2 \cdot \pi}{\omega} \tag{8}$$

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Damped unexcited harmonic oscillator - 1st lecture (2020.02.26)

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Unexcited vibration of a harmonic oscillator

System scheme

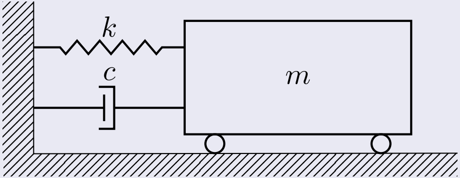

Figure 6: A damped harmonic oscillator

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Harmonic oscillator

Governing equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{9}$$

where:

- m - mass,
- c - damping constant,
- k - stiffness coefficient.

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Unexcited vibration of a harmonic oscillator

Simplified governing equation

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{10}$$

For the further computations there is assumed that:

$$\frac{c}{m} = 2 \cdot h, \quad \frac{k}{m} = \omega_0^2 \tag{11}$$

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Solution of the governing equation

Governing equation expressed in acceleration

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x = 0 \tag{12}$$

Predicted form of the solution

$$x(t) = Ce^{rt} \tag{13}$$

Substitution of the predicted solution (13) leads to the following equation:

$$\left(r^2 + 2hr^1 + \omega_0^2 r^0\right) Ce^{rt} = 0 \tag{14}$$

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Solution of the governing equation

Characteristic polynomial

$$\underbrace{1}_{A} r^2 + \underbrace{2h}_{B} r^1 + \underbrace{\omega_0^2}_{C} r^0 = 0 \qquad (15)$$

Roots of the characteristic polynomial are as follows:

$$r_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad (16)$$

$$r_{1,2} = \frac{-2h \pm \sqrt{4h^2 - 4\omega_0^2}}{2} = -h \pm \sqrt{h^2 - \omega_0^2} \qquad (17)$$

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Solution of the governing equation

Final result

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{18}$$

There are 3 cases:

- overdamped vibration - if $h^2 > \omega_0^2$;
- critically-damped vibration - if $h^2 = \omega_0^2$;
- underdamped vibration - if $h^2 < \omega_0^2$.

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Solution of the governing equation

Overdamped oscillator

$$h^2 > \omega_0^2 \tag{19}$$

The solution has a following form:

$$x(t) = C_1 e^{-ht-\sqrt{h^2-\omega_0^2}t} + C_2 e^{-ht+\sqrt{h^2-\omega_0^2}t} \tag{20}$$

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Solution of the governing equation

Critically-damped oscillator

$$h^2 = \omega_0^2 \tag{21}$$

The solution has a following form:

$$x(t) = C_1 e^{-ht} + C_2 t e^{-ht} \tag{22}$$

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Solution of the governing equation

Underdamped oscillator

$$h^2 < \omega_0^2 \tag{23}$$

The solution has a following form:

$$x(t) = C_1 e^{-ht - \mathrm{i}\sqrt{\omega_0^2 - h^2} t} + C_2 e^{-ht + \mathrm{i}\sqrt{\omega_0^2 - h^2} t} \tag{24}$$

For further simplification of the calculations the following symbol is introduced:

$$\omega_h = \sqrt{\omega_0^2 - h^2} \tag{25}$$

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Solution of the governing equation

Underdamped oscillator

$$h^2 < \omega_0^2 \tag{26}$$

The rearranged solution has a following form:

$$x(t) = C_1 e^{-ht - \mathrm{i}\omega_h t} + C_2 e^{-ht + \mathrm{i}\omega_h t} \tag{27}$$

Application of Euler's formula results in the oscillatory form of the solution:

$$x(t) = e^{-ht} \left(D_1 \cos \omega_h t + D_2 \sin \omega_h t \right) \tag{28}$$

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Damped free vibration - 1st test (2020.02.26)

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First test

Problem

Solve the governing equation of the system depicted in the figure 7. Determine its natural frequency of a damped vibration and the damping ratio.

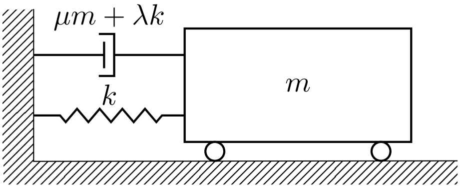

Figure 7: A damped harmonic oscillator

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Damped free vibration - 2nd lecture (2020.03.04)

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Damping in unexcited harmonic oscillator

Time solutions for the different damping types

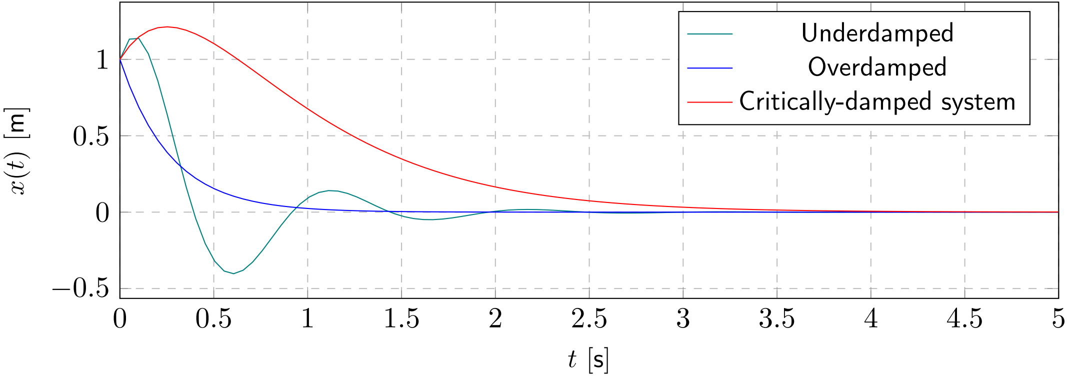

Figure 8: Time response of damped harmocnic oscillator

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Harmonic synthesis - 2nd lecture (2020.03.04)

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Harmonic synthesis

Determine the new harmonic signal composed of the two signals x_1, x_2 being a sum, where: $x_1(t) = A \cdot \sin(\omega t)$ and $x_2(t) = B \cdot \cos(\omega t)$

Solution

$$x(t) = x_1(t) + x_2(t) = a \sin(\omega t + \varphi) \tag{29}$$

$$A \cdot \sin(\omega t) + B \cdot \cos(\omega t) = a \cos(\varphi) \cdot \sin(\omega t) + a \sin(\varphi) \cdot \cos(\omega t) \tag{30}$$

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Solution

Compering the corresponding factors of the left and right hand side of the equation (30) one might obtained:

$$A = a\cos(\varphi), \quad B = a\sin(\varphi) \tag{31}$$

Rearranging the equation (31) for the Pythagorean identity, the amplitude a might be obtained:

$$A^2 + B^2 = a^2(\cos^2(\varphi) + \sin^2(\varphi)) \tag{32}$$

$$a^2 = A^2 + B^2 \tag{33}$$

$$a = \sqrt{A^2 + B^2} \tag{34}$$

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Solution

Rearranging the equation (31) for another trigonometric identity, the phase shift φ might be obtained:

$$\frac{B}{A} = \frac{\sin(\varphi)}{\cos(\varphi)} = \operatorname{tg}(\varphi) \tag{35}$$

$$\varphi = \operatorname{arc\,tg}\left(\frac{B}{A}\right) \tag{36}$$

The new signal might be then expressed as follows:

$$x(t) = \sqrt{A^2 + B^2}\sin\left(\omega t + \operatorname{arc\,tg}\left(\frac{B}{A}\right)\right) \tag{37}$$

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Logarithmic decrement - 2nd lecture (2020.03.04)

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Decrement of damping

Definition

$$\Delta = \frac{A_n}{A_{n+1}} \tag{38}$$

Relationship with the system parameters

$$\Delta = \frac{x(t_0)}{x(t_0 + T_h)} = \frac{e^{-ht_0}\left(D_1 \cos \omega_h t_0 + D_2 \sin \omega_h t_0\right)}{e^{-h(t_0+T_h)}\left(D_1 \cos \omega_h (t_0 + T_h) + D_2 \sin \omega_h (t_0 + T_h)\right)} \tag{39}$$

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Decrement of damping

Relationship with the system parameters

$$\Delta = e^{hT_h} \tag{40}$$

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